Algebraic Properties of the Multistate Population Matrix Model

1st Sisilia Sylviani   
*department of mathemtics   
Universitas Padjadjaran*Sumedang, Indonesia  
sisilia.sylviani@unpad.ac.id2nd Ema Carnia  
*department of mathemtics   
Universitas Padjadjaran*Sumedang, Indonesia  
ema.carnia@unpad.ac.id3rd A.K. Supriatna  
*department of mathemtics   
Universitas Padjadjaran*Sumedang, Indonesia  
a.k.supriatna@unpad.ac.id

*Abstract*—Discrete time population growth is often modeled by a matrix. Many growth parameters such as growth rate, reproduction rate, as well as the movement of the population are easily included in a matrix model. This paper will discuss a matrix model that describes the dynamics of a population having some stages of life and occupying some different patches. The matrix, which is the product of two matrices S and D, is often called SD matrix. The matrix S is a diagonal block matrix in which its block is a sub-stochastic column matrix. The matrix S represents the movement of a population between locations (patches). On the other hand, the matrix D is a block matrix in which its block is a nonnegative real diagonal matrix. The matrix D describes the population growth in specific patches. The paper will focus on the properties of the SD matrix from the algebraic point of view, particularly the spectral radius of the matrix. It will be shown that the spectral radius of the SD matrix is less than the spectral radius of D meanwhile the condition does not hold for the block matrices SD and D.

Keywords— matrix, population, spectral radius

# Introduction

One way to build a multistate population model is to use the **SD** matrix model.The matrix **S** is a diagonal block matrix in which its block is a sub-stochastic column matrix.

Let be square matrix with nonnegative entries. The matrix defined as a block diagonal matrix with diagonal blocks or can be written as follow

(1)

An matrix with non-negative entries is column sub stochastic if all its columns sum are less than or equal to +1, or can be written as follows [4]

(2)

where

The matrix describes the movement of a population between patches and can be written as follow

(3)

where is substochastic matrices [4]. On the other hand, let be a permutation matrix and be an nonnegative matrix representing the growth dynamic in location j, then the matrix is a block matrix in which its block is a diagonal matrix with nonnegative real entries and can be written as follow

. (4)

The multiplication of and is also a matrix model that represent a biological model that account for spatial structure. The discussion in this paper is focus on the study of spectral radius properties of the matrix model.

Let be a square matrix, the spectral radius of is the non-negative number which is defined as follow [3]

where is eigenvalues for .

Another term that is used in this paper is the Column Sum Norm. The Column Sum Norm of a square matrix is defined as follow [4]

(5)

.

# MAIN RESULTS

The spectral radius of a matrix is used to determine the rate of growth of a population that described in a matrix model. In this section it will be shown that the spectral radius of matrix model (where is a column substocasthic matrix and is a diagonal nonnegative matrx) is less than or equal to the spectral radius of . However, before we proceed it will be explained first some theorem that will be used to proof that properties.

**Theorem 1** [3] The largest eigenvalue of a substochastic matrix is 1 or in the other words is that the radius spectral of a substochastic matrix is equal to 1.

**Lemma 2** [4] Let be a square matrix, then

Proof.

Let be an arbitrary matrix. Let be eigenvector of such that

where is the eigenvalue of that correspond with and , then we have

.

Since is scalar, then based on norm properties, we obtain

and then we have

Hence

is true for all thus

The Lemma Above implies the following Proposition.

**Proposition 3** [3] Let be a diagonal nonnegative matrix and a column substochastic matrix. Then

Proof. Based on the lemma above we have

Since the S is column sub stochastic matrix, then all of the columns sum are less than or equal to +1. Furthermore, the spectral radius of D is equal to its column sum norm. Thus, we have

It prove the lemma.

Propositions 3 has some important meanings, especially for biological models that account for spatial structure [3]. One of them is, from that fact one can know whether population growth occurs before dispersal or population growth occurs after dispersal.

However, the conditions cannot always apply for matrix model. Suppose where is substochastic matrices then for every block matrix we can not always have condition of .

For example let and and Then, from (4) we obtain

and we also have .

Meanwhile, Let and , then =

Multiplying ***S*** and ***D*** we have

and we have ρ. In other words . Hence, in general we can't always have .

# CONCLUSION

The spectral radius from SD matrix model is less than or equal to the spectral radius of the matrix D. That relation has some important meanings, especially for biological model that account for spatial structure [4]. One of them is that one can know whether population growth occurs before dispersal or after dispersal. However for matrix model, the condition does not always apply.

##### Acknowledgment

This research is funded by the Academic Leadership Grant (2015-2019) of Universitas Padjadjaran.

##### References

1. Caswell, H., Matrix Population Models: Construction, Analysis and Interpretation, Second Edition. Sunderland:Sinauer Associates, Inc., 2001.
2. Diekmann, O., J. A. P., Heesterbeek., Mathematical Epidemiology of Infectious Disease, Wiley Series in Mathematical and Computational Biology, Chichester: John Wiley \& Sons Ltd, 2000.
3. Horn, R. A., Johnson, C. R., Matrix Analysis, Cambridge: University Press Cambridge,1985.
4. Li, Chi-Kwong, Schreiber, S. J., On Dispersal and Population Growth for Multistate Matrix Models, Linear Algebra and Its Applications, 418: 900-912, 2006.
5. Strang, G., *Linear Algebra and Its Applications*, Thomson Brooks/cole, 2006.
6. Sylviani, S., Carnia, E. Supriatna, A.K., An Algebraic Problem Arising in Biomathematics, Jurnal Teknologi pp 75–79, 2016.